Periodic Stability for Nonlinear Systems Generated by Time-Dependent Subdifferentials

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Periodic Stability for Nonlinear Systems Generated by Time-Dependent Subdifferentials

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Abstract. A nonlinear time periodic system, which is governed by time-dependent subdifferentials, is considered in a (real) Hilbert space. Recent results on global attractors for our system are presented. Also, these abstract results are applied to a phase-field model with constraint of the Penrose-Fife type.

1. Introduction

Let us consider a nonlinear evolution system

$$(\mathrm{P})_s \qquad u'(t) + \partial\varphi^t(u(t)) + g(t, u(t)) \ni f(t), \quad t > s (\geq 0), \tag{1.1}$$

which is governed by the subdifferential $\partial\varphi^t$ of a time-dependent proper, l.s.c. convex function φ^t on a (real) Hilbert space H, where $u' = \frac{du}{dt}$, $g(t,\cdot)$ is a perturbation and f is a forcing term. In this paper, assuming that $\varphi^t(\cdot)$, $g(t,\cdot)$ and $f(t)$ are periodic in time t with a common period $T_0 (> 0)$, we investigate the asymptotic behaviour of the dynamical process (evolution operator) $U(t,s) : \overline{D(\varphi^s)} \to \overline{D(\varphi^t)}$, $0 \leq s \leq t < +\infty$, associated to system $(P)_s$; in fact, we present that $(\mathrm{P})_s$ has at least one time-periodic solution with period T_0 and for each $\tau \in R_+ := [0, +\infty)$ the discrete dynamical process $\{T_\tau^n\}_{n=1}^\infty$ on $\overline{D(\varphi^\tau)}$, generated by $T_\tau := U(T_0 + \tau, \tau)$, possesses a global attractor $\mathcal{A}_\tau$ which is periodic in τ with period T_0.

We recall some works (cf. [2]) treating similar topics for a class of semilinear evolution equations.

As an application of our abstract results we treat the large time behaviour of a phase-field model with constraint of the Penrose-Fife type, which is a system of nonlinear PDEs as follows:

$$[\theta+\lambda(t,x,w)]_t-\Delta\left(-\frac{1}{\theta}+\mu\theta\right)=q(t,x) \quad \text{in } Q_s:=(s,+\infty)\times\Omega,\ s\geq 0, \quad (1.2)$$

$$w_t-\kappa\Delta w+\beta(w)+\sigma(w)+\frac{\lambda_w(t,x,w)}{\theta}\ni 0 \quad \text{in } Q_s, \quad (1.3)$$

with boundary conditions

$$\frac{\partial}{\partial n}\left(-\frac{1}{\theta}+\mu\theta\right)+n_0\left(-\frac{1}{\theta}+\mu\theta\right)=h(t,x), \quad \text{on } \Sigma_s:=(s,+\infty)\times\Gamma, \quad (1.4)$$

$$\frac{\partial w}{\partial n}=0 \quad \text{on } \Sigma_s. \quad (1.5)$$

Here Ω is a bounded domain in $R^N, 1\leq N\leq 3$, with smooth boundary $\Gamma:=\partial\Omega$; $\beta(\cdot)$ is a maximal monotone graph in $R\times R$; λ is a smooth function on $R_+\times\Omega\times R$, convex in $w\in R$ and periodic in t with period T_0; $\sigma(\cdot)$ is a smooth function on R; n_0, κ and μ are positive constants and q, h are given data.

The phase-field models with constraint were earlier studied in [5, 10, 13]. In [3, 4], the existence and uniqueness result for the Cauchy problem of the system (1.2)-(1.5) was obtained for good initial data θ_0 and w_0 in the case of $\lambda(t,x,w)=\lambda(x,w)$ without convexity assumption with respect to w.

Notation. Throughout this paper, let H be a (real) Hilbert space with norm $|\cdot|_H$ and inner product $(\cdot,\cdot)_H$. For a proper l.s.c. convex function φ on H we denote by $D(\varphi)$ and $\partial\varphi$ the effective domain and subdifferential of φ, respectively; the domain and range of $\partial\varphi$ are denoted by $D(\partial\varphi)$ and $R(\partial\varphi)$, respectively. We refer for fundamental properties of subdifferentials to [1].

When a given function is periodic in time with period T_0, we say simply that the function is T_0-periodic.

For a point z in H and non-empty subsets X and Y of H, we define

$$\mathrm{dist}_H(z,Y):=\inf_{y\in Y}|z-y|_H, \quad \mathrm{dist}_H(X,Y):=\sup_{x\in X}\inf_{y\in Y}|x-y|_H.$$

2. Abstract results (existence of a T_0-periodic solution)

Evolution equation $(\mathrm{P})_s$ is formulated for any family $\{\varphi^t\}$ in the class $\Phi_p(\{a_r\},\{b_r\};T_0)$ specified below, where $\{a_r\}:=\{a_r;r\geq 0\}$ and $\{b_r\}:=$

$\{b_r; r \geq 0\}$ are families of real functions in $W^{1,2}_{loc}(R_+)$ and $W^{1,1}_{loc}(R_+)$, respectively, such that

$$\sup_{t\geq 0} |a'_r|_{L^2(t,t+1)} + \sup_{t\geq 0} |b'_r|_{L^1(t,t+1)} < +\infty \quad \text{for every } r \geq 0.$$

Definition 2.1. $\{\varphi^t\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$ if and only if φ^t is a proper l.s.c. convex function on H such that

$$\varphi^{t+T_0}(\cdot) = \varphi^t(\cdot) \quad \text{on } H, \quad \forall t \in R_+,$$

$\{z \in H; |z|_H \leq k,\ \varphi^t(z) \leq k\}$ is compact in H for every $t \geq 0$ and $k \geq 0$,

and the following property $(*)$ is fulfilled:

$(*)$ For each $r \in R_+$, s, $t \in R_+$ and $z \in D(\varphi^s)$ with $|z|_H \leq r$, there exists $\tilde{z} \in D(\varphi^t)$ such that

$$|\tilde{z} - z|_H \leq |a_r(t) - a_r(s)|(1 + |\varphi^s(z)|^{\frac{1}{2}})$$

and

$$\varphi^t(\tilde{z}) - \varphi^s(z) \leq |b_r(t) - b_r(s)|(1 + |\varphi^s(z)|).$$

Next, we introduce the class $\mathcal{G}_p(\{\varphi^t\}; T_0)$ associated with $\{\varphi^t\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$.

Definition 2.2. $\{g(t,\cdot)\} \in \mathcal{G}_p(\{\varphi^t\}; T_0)$ if and only if $g(t,\cdot)$ is an operator from H into H which fulfills the following conditions (g1)-(g6):

(g1) $D(\varphi^t) \subset D(g(t,\cdot)) \subset H$ for all $t \in R_+$ and $g(\cdot, v(\cdot))$ is (strongly) measurable on J for any interval $J \subset R_+$ and $v \in L^2_{loc}(J; H)$ with $v(t) \in D(\varphi^t)$ for a.e. $t \in J$.

(g2) There are positive constants C_0, C_1 and C_2 such that

$$|g(t,z)|^2_H \leq C_0\varphi^t(z) + C_1|z|^2_H + C_2, \quad \forall t \in R_+, \quad \forall z \in D(\varphi^t).$$

(g3) (Demi-closedness) If $\{t_n\} \subset R_+$, $\{z_n\} \subset H$, $t_n \to t$, $z_n \to z$ in H (as $n \to +\infty$) and $\{\varphi^{t_n}(z_n)\}$ is bounded, then $g(t_n, z_n) \to g(t,z)$ weakly in H.

(g4) For each $\varepsilon > 0$, there exists a positive constant $C_\varepsilon > 0$ such that

$$|(g(t,z_1) - g(t,z_2), z_1 - z_2)_H| \leq \varepsilon(z_1^* - z_2^*, z_1 - z_2)_H + C_\varepsilon |z_1 - z_2|_H^2,$$

$$\forall t \in R_+,\ \forall z_i \in D(\varphi^t),\ \forall z_i^* \in \partial\varphi^t(z_i),\ i = 1,2.$$

(g5) (Coerciveness) For each bounded set B in H there are positive constants $C_0(B)$ and $C_1(B)$ such that

$$\varphi^t(z) + (g(t,z), z - b)_H \geq C_0(B)|z|_H^2 - C_1(B),$$

$$\forall t \in R_+,\ \forall z \in D(\varphi^t), \forall b \in B.$$

(g6) (T_0-periodicity) $g(t + T_0, \cdot) = g(t, \cdot)$ on H, $\forall t \in R_+$.

The notion of a solution of $(\mathrm{P})_s$ is given in the next definition.

Definition 2.3. (1) A function $u : [s,T] \to H$, $0 \leq s < T < +\infty$, is a solution of $(\mathrm{P})_s$ on $[s,T]$, if $u \in C([s,T];H) \cap W^{1,2}_{loc}((s,T];H)$, $\varphi^{(\cdot)}(u(\cdot)) \in L^1(s,T)$, $g(\cdot, u(\cdot)) \in L^2(s,T;H)$ and

$$f(t) - u'(t) - g(t,u(t)) \in \partial\varphi^t(u(t)) \quad \text{for a.e. } t \in [s,T].$$

A function u is called a solution of $(\mathrm{P})_s$ on $[s,+\infty)$, if it is a solution of $(\mathrm{P})_s$ on $[s,T]$ for every finite $T > s$. Also, $u : [s,T]$ or $[s,+\infty) \to H$ is called a solution of the Cauchy problem for $(\mathrm{P})_s$ with initial value $u_0 \in H$, if it is a solution of $(\mathrm{P})_s$ and $u(s) = u_0$.
(2) u is called a T_0-periodic solution of $(\mathrm{P})_s$ on $[s,+\infty)$, $s \geq 0$, if u is a solution of $(\mathrm{P})_s$ which satisfies T_0-periodicity condition:

$$u(t) = u(t + T_0) \qquad \text{for any } t \in [s,+\infty).$$

Theorem 2.1. (cf. [14; Theorem 2.1.]) *Assume that* $\{\varphi^t\} \in \Phi_p(\{a_r\},\{b_r\};T_0)$, $\{g(t,\cdot)\} \in \mathcal{G}_p(\{\varphi^t\};T_0)$ *and* $f \in L^2_{loc}(R_+;H)$. *Then, the Cauchy problem for* $(P)_s$, $s \geq 0$, *has one and only one solution* u *on* $J_s := [s,+\infty)$ *such that* $(\cdot - s)^{\frac{1}{2}}u'(\cdot) \in L^2_{loc}(J_s;H)$, $(\cdot - s)\varphi^{(\cdot)}(u(\cdot)) \in L^\infty_{loc}(J_s)$ *and* $\varphi^{(\cdot)}(u(\cdot))$ *is absolutely continuous on any compact subinterval of* $(s,+\infty)$, *provided that* $u_0 \in \overline{D(\varphi^s)}$. *In particular, if* $u_0 \in D(\varphi^s)$, *then the solution* u *satisfies that* $u' \in L^2_{loc}(J_s;H)$ *and* $\varphi^{(\cdot)}(u(\cdot))$ *is absolutely continuous on any compact interval in* J_s.

Based on this existence result, we can define the solution operator (dynamical process) associated to $(\mathrm{P})_s$.

Definition 2.4. For every $0 \le s \le t < +\infty$ we denote by $U(t,s)$ the mapping from $\overline{D(\varphi^s)}$ into $\overline{D(\varphi^t)}$ which assigns to each $u_0 \in \overline{D(\varphi^s)}$ the element $u(t) \in \overline{D(\varphi^t)}$, where u is the unique solution of $(P)_s$ with initial condition $u(s) = u_0$.

It is easy to check the following properties of $\{U(t,s)\} := \{U(t,s); 0 \le s \le t < +\infty\}$:

(U1) $U(s,s) = I$ on $\overline{D(\varphi^s)}$ for any $s \in R_+$;

(U2) $U(t_2, s) = U(t_2, t_1) \circ U(t_1, s)$ for any $0 \le s \le t_1 \le t_2 < +\infty$;

(U3) $U(t + T_0, s + T_0) = U(t,s)$ for any $0 \le s \le t < +\infty$, that is, U is T_0-periodic.

In terms of $U(t,s)$, global estimates of solutions for $(\mathrm{P})_s$ are stated as follows:

Theorem 2.2. (cf. [14; Theorem 2.2]) (Global boundedness of the solution for $(\mathrm{P})_s$) *In addition to all the assumptions of Theorem 2.1, suppose that*

$$S_f := \sup_{t \ge 0} |f|_{L^2(t,t+1;H)} < +\infty.$$

Then, for any bounded set B in H,

(i) *There is a positive constant $R_1 := R_1(S_f, B)$ such that*

$$|U(t,s)z|_H \le R_1 \quad \text{for any } t \ge s \ge 0 \text{ and all } z \in \overline{D(\varphi^s)} \cap B.$$

(ii) *There is a positive constant $R_2 := R_2(S_f, B)$ such that*

$$\int_t^{t+1} |\varphi^\tau(U(\tau,s)z)| d\tau \le R_2 \quad \text{for all } t \ge s \ge 0 \text{ and } z \in \overline{D(\varphi^s)} \cap B.$$

(iii) *For each $\delta > 0$, there is a positive constant $R_3 := R_3(S_f, B, \delta)$ such that*

$$|\varphi^t(U(t,s)z)| + \left| \frac{d}{dt} U(\cdot, s) z \right|^2_{L^2(t,t+1;H)} \le R_3,$$

for all $s \ge 0$, $t \ge s + \delta$ and $z \in \overline{D(\varphi^s)} \cap B$.

With the help of global estimates mentioned in Theorem 2.2 as well as a convergence result [14; Lemma 4.1] we can prove:

Theorem 2.3 *Assume that the same assumptions are made as in Theorem 2.1 and* $f \in L^2_{loc}(R_+; H)$ *is* T_0*-periodic, i.e.*

$$f(t) = f(t + T_0) \qquad \text{for any } t \in R_+.$$

Then for each $s \in R_+$*, there exists a* T_0*-periodic solution* u *for* $(P)_s$.

In the proof of Theorem 2.3, the crucial step is to show that the mapping $T_s := U(T_0 + s, s) : \overline{D(\varphi^s)} \to \overline{D(\varphi^{s+T_0})} = \overline{D(\varphi^s)}$ has a fixed point, which can be done by the Schauder's fixed point theorem. See [9] for a detailed proof.

3. Abstract results (global attractors)

In this section, we present some recent results on global attractors for the solution operator $U(t, s)$ associated with $(\mathrm{P})_s$; all the assumptions of Theorem 2.1 are made as well.

For each $\tau \geq 0$ we define a mapping T_τ by putting

$$T_\tau := U(T_0 + \tau, \tau) : \overline{D(\varphi^\tau)} \to \overline{D(\varphi^\tau)},$$

and its k-th iteration by

$$T_\tau^k := T_\tau \circ T_\tau \circ \cdots \circ T_\tau, \ k = 0, 1, 2, \cdots.$$

Essentially using the theory of discrete dynamical systems (cf. [7, 15]), we have:

Theorem 3.1. *Assume that* $\{\varphi^t\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$, $\{g(t, \cdot)\} \in \mathcal{G}_p(\{\varphi^t\}; T_0)$, $f \in L^2_{loc}(R_+; H)$ *is* T_0*-periodic. Then, for each* $\tau \geq 0$*, there exists a subset* $\mathcal{A}_\tau$ *of* $D(\varphi^\tau)$ *such that*

(i) $\mathcal{A}_\tau$ *is non-empty, compact and connected in* H*,*

(ii) $T_\tau^k \mathcal{A}_\tau = \mathcal{A}_\tau$ *for all* $k = 0, 1, 2, \cdots$*,*

(iii) *for each bounded set* B *in* H *and each number* $\epsilon > 0$ *there exists a positive integer* $N_{B,\epsilon}$ *such that*

$$dist_H(T_\tau^k z, \mathcal{A}_\tau) < \epsilon, \quad \forall z \in \overline{D(\varphi^\tau)} \cap B, \ \forall k \geq N_{B,\epsilon}.$$

Moreover, for any $0 \le s \le \tau < +\infty$,

$$\mathcal{A}_\tau = U(\tau, s)\mathcal{A}_s \tag{3.1}$$

holds.

Remark 3.1. (1) For any $\tau \ge 0$, choose $m_\tau \in Z_+$ and $\sigma_\tau \in [0, T_0)$ so that $\tau = \sigma_\tau + m_\tau T_0$. Then, Theorem 3.1 (ii) implies that $\mathcal{A}_\tau = \mathcal{A}_{\sigma_\tau}$, hence the set-valued mapping $\tau \to \mathcal{A}_\tau$ is T_0-periodic.

(2) In [11, 12], periodic system $(\mathrm{P})_s$ with $g \equiv 0$ was studied, and it was shown that some solutions do not approach to any periodic solutions as $t \to +\infty$; in other words the asymptotic behaviour (as $t \to +\infty$) along a single solution is not periodic in time. However, as was seen in (1), the global attractor $\mathcal{A}_\tau$ is T_0-periodic.

(3) Relation (3.1) of Theorem 3.1 implies that $U(\tau, s)$ is a topological mapping from $\mathcal{A}_s$ onto $\mathcal{A}_\tau$.

4. Application to a phase-field model with constraint

In this section, let us consider the periodic problem $(\mathrm{PFC})_s$ of a phase-field model with constraint for the Penrose-Fife type:

$$(\mathrm{PFC})_s \begin{cases} [\theta + \lambda(t,x,w)]_t - \Delta\left(-\dfrac{1}{\theta} + \mu\theta\right) = q(t,x) & \text{in } Q_s, \\ w_t - \kappa\Delta w + \beta(w) + \sigma(w) + \dfrac{\lambda_w(t,x,w)}{\theta} \ni 0 & \text{in } Q_s, \\ \dfrac{\partial}{\partial n}\left(-\dfrac{1}{\theta} + \mu\theta\right) + n_0\left(-\dfrac{1}{\theta} + \mu\theta\right) = h(t,x) & \text{on } \Sigma_s, \\ \dfrac{\partial w}{\partial n} = 0 \quad \text{on } \Sigma_s, \end{cases}$$

under the same notation as section 1.

We assume precisely that

- λ is a smooth function on $R_+ \times R^N \times R$ such that $\lambda(t,x,w)$ is convex with respect to $w \in R$ for each $(t,x) \in R_+ \times R^N$ and is T_0-periodic for each $(x,w) \in \Omega \times R$;

- λ and its partial derivatives $\lambda_w := \dfrac{\partial\lambda}{\partial w}$, $\lambda_t := \dfrac{\partial\lambda}{\partial t}$ are bounded on $R_+ \times \overline{\Omega} \times [-1,1]$, namely,

$$L_\lambda := \sup\left\{|\lambda(t,x,w)| + |\lambda_w(t,x,w)| + |\lambda_t(t,x,w)|\ ;\right.$$

$$x \in \overline{\Omega},\ t \geq 0,\ |w| \leq 1 \Big\} < +\infty;$$

- β is a maximal monotone graph in $R \times R$ such that $\overline{D(\beta)} = [-1,1]$; we fix a proper l.s.c. convex and non-negative function $\hat{\beta}$ on R whose subdifferential $\partial\hat{\beta}$ coincides with β in R;

- σ is a smooth function on R;

- n_0, μ and κ are positive constants;

- $f \in L^2_{loc}(R_+; L^2(\Omega))$ and $h \in L^2_{loc}(R_+; L^2(\Gamma))$ are T_0-periodic in time.

We need some notation in order to reformulate $(\mathrm{PFC})_s$ as an evolution equation in terms of subdifferential.

Let V be the Sobolev space $H^1(\Omega)$ with norm

$$|v|_V := \left\{ \int_\Omega |\nabla v|^2 dx + n_0 \int_\Gamma |v|^2 d\Gamma \right\}^{\frac{1}{2}}, \qquad \forall v \in V,$$

V^* be the dual space of V and F be the duality mapping from V onto V^*, namely,

$$\langle Fv, z \rangle := \int_\Omega \nabla v \cdot \nabla z dx + n_0 \int_\Gamma vz d\Gamma, \quad \forall v, \forall z \in V,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V^* and V.

Given $q \in L^2(\Omega)$ and $h \in L^2(\Gamma)$, an element $q^* \in V^*$ is uniquely determined by

$$\langle q^*, z \rangle := \int_\Omega qz dx + \int_\Gamma hz d\Gamma, \quad \forall z \in V,$$

and it is easy to check that $Fv = q^*$ is formally equivalent to

$$-\Delta v = q \text{ in } \Omega, \quad \frac{\partial v}{\partial n} + n_0 v = h \text{ on } \Gamma; \tag{4.1}$$

in fact, (4.1) is satisfied in the variational sense that

$$\int_\Omega \nabla v \cdot \nabla z dx + n_0 \int_\Gamma vz d\Gamma = \int_\Omega qz dx + \int_\Gamma hz d\Gamma\ (= \langle q^*, z \rangle), \quad \forall z \in V.$$

By notation Δ_N we denote the Laplacian, with homogeneous Neumann boundary condition, in $L^2(\Omega)$, more precisely,

$$D(\Delta_N) = \left\{\ z \in H^2(\Omega)\ \middle|\ \frac{\partial z}{\partial n} = 0 \text{ in } H^{\frac{1}{2}}(\Gamma)\ \right\}$$

and

$$\Delta_N z = \Delta z \text{ a.e. in } \Omega \text{ for any } z \in D(\Delta_N).$$

It is well known that $-\Delta_N$ is singlevalued and maximal monotone in $L^2(\Omega)$.

As was seen in the recent paper [6], we can reformulate $(\text{PFC})_s$ as an evolution equation with a new variable $e := \theta + \lambda(\cdot,\cdot,w)$, in the following form:

$$\frac{d}{dt}\begin{pmatrix} e(t) \\ w(t) \end{pmatrix} + \begin{pmatrix} F(\alpha(e(t) - \lambda(t,\cdot,w(t))) + \mu e(t)) \\ -\kappa\Delta_N w(t) + \xi(t) - \alpha(e(t) - \lambda(t,\cdot,w(t)))\lambda_w(t,\cdot,w(t)) \end{pmatrix}$$
$$+ \begin{pmatrix} -\mu F\lambda(t,\cdot,w(t)) \\ \sigma(w(t)) \end{pmatrix} = \begin{pmatrix} q^*(t) \\ 0 \end{pmatrix}, \tag{4.2}$$

in the product space

$$H := \begin{matrix} V^* \\ \times \\ L^2(\Omega) \end{matrix},$$

where H is a Hilbert space with inner product $(\cdot,\cdot)_H$ given by

$$(U_1, U_2)_H := \left\langle e_1, F^{-1}e_2 \right\rangle + \int_\Omega w_1 w_2 dx,$$

for all $U_i := \begin{pmatrix} e_i \\ w_i \end{pmatrix} \in H$ $(i = 1, 2)$, $q^*(t)$ is the element of V^* determined by

$$\langle q^*(t), z\rangle = \int_\Omega q(t) z dx + \int_\Gamma h(t) z d\Gamma, \quad \forall z \in V,$$

and $\alpha(r) := -\dfrac{1}{r}$ for $r > 0$.

Let us define φ^t on H by putting

$$\varphi^t(u) := \begin{cases} \displaystyle\int_\Omega \left\{ -\log(e - \lambda(t,\cdot,w)) + \frac{\mu}{2}|e|^2 \right\} dx + \frac{\kappa}{2}\int_\Omega |\nabla w|^2 dx + \int_\Omega \hat{\beta}(w) dx \\ \qquad \text{if } u := \begin{pmatrix} e \\ w \end{pmatrix} \in \begin{matrix} L^2(\Omega) \\ \times \\ H^1(\Omega) \end{matrix} \\ \qquad \text{with } \log(e - \lambda(t,\cdot,w)) \in L^1(\Omega),\ \hat{\beta}(w) \in L^1(\Omega), \\ +\infty \quad \text{otherwise.} \end{cases}$$

According to the result of [6, 14], we have the following lemmas.

Lemma 4.1. *(1) For each $t \in R_+$, φ^t is proper l.s.c. convex on H and T_0-periodic, and* $D(\varphi^t) \subset \begin{matrix} L^2(\Omega) \\ \times \\ H^1(\Omega) \end{matrix}$. *Moreover, there are positive constants ν_0, ν_1, independent of $t \in R_+$, such that*

$$\varphi^t(u) \geq \nu_0(|e|^2_{L^2(\Omega)} + |w|^2_{H^1(\Omega)}) - \nu_1, \quad \forall u := \begin{pmatrix} e \\ w \end{pmatrix} \in D(\varphi^t).$$

(2) $\{\varphi^t\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$, where $a_r(t) = b_r(t) := R_0 t$ for all $r \geq 0$ and $t \in R_+$, with a (sufficiently large) constant $R_0 > 0$; in fact, we can choose as R_0 a constant of the form const.L_λ.

Lemma 4.2. *For each $t \in R_+$,*

$$D(\partial\varphi^t) = \left\{ \begin{pmatrix} e \\ w \end{pmatrix} \in \begin{matrix} L^2(\Omega) \\ \times \\ H^2(\Omega) \end{matrix} ; \begin{matrix} \alpha(e - \lambda(t,\cdot,w)) + \mu e \in V, \ \dfrac{\partial w}{\partial n} = 0 \ \mathit{in} \ H^{\frac{1}{2}}(\Gamma), \\ \exists \xi \in L^2(\Omega) \ \mathit{such\ that} \ \xi \in \beta(w) \ \mathit{a.e.\ on} \ \Omega \end{matrix} \right\}$$

and if $\begin{pmatrix} e^* \\ w^* \end{pmatrix} \in \partial\varphi^t \begin{pmatrix} e \\ w \end{pmatrix}$, *then*

$$\begin{aligned} e^* &= F\left(\alpha(e - \lambda(t,\cdot,w)) + \mu e\right), \\ w^* &= -\kappa\Delta_N w + \xi - \alpha(e - \lambda(t,\cdot,w))\lambda_w(t,\cdot,w) \\ &\mathit{for\ some}\ \xi \in L^2(\Omega)\ \mathit{such\ that}\ \xi \in \beta(w)\ \mathit{a.e.\ on}\ \Omega. \end{aligned} \tag{4.3}$$

Moreover, we have

$$(u_1^* - u_2^*, u_1 - u_2)_H \geq \mu|e_1 - e_2|^2_{L^2(\Omega)} + \kappa|\nabla(w_1 - w_2)|^2_{L^2(\Omega)} \tag{4.4}$$

$$\forall t \in R_+, \ \forall u_i := \begin{pmatrix} e_i \\ w_i \end{pmatrix} \in D(\partial\varphi^t), \ \forall u_I^* \in \partial\varphi^t(u_i), \ i = 1, 2.$$

Now, combining expressions (4.2) and (4.3), we see that our system $(\text{PFC})_s$ is reformulated as the evolution equation

$$u'(t) + \partial\varphi^t(u(t)) + g(t, u(t)) \ni f(t) \quad \text{in } H, \ t > s(\geq 0),$$

where

$$g(t,u) := \begin{pmatrix} -\mu F\lambda(t,\cdot,w) \\ \sigma(w) \end{pmatrix} \text{ for } u := \begin{pmatrix} e \\ w \end{pmatrix} \in \begin{matrix} L^2(\Omega) \\ \times \\ H^1(\Omega) \end{matrix}, \ f(t) := \begin{pmatrix} q^*(t) \\ 0 \end{pmatrix}. \tag{4.5}$$

It is not difficult to check with the help of (4.4) that the operator $g(t,\cdot)$ defined by (4.5) satisfies all the conditions (g1)-(g6) in Definition 2.2.

As direct consequences of Theorems 2.3 and 3.1, we see that the periodic system (4.1)-(4.4) has at least one T_0-periodic solution and the global attractor $\mathcal{A}_\tau$ for each $\tau \geq 0$. Namely, for any bounded subset $B \in X$ any solution $[\theta(nT_0+\tau)+\lambda(nT_0+\tau,\cdot,w(nT_0+\tau)), w(nT_0+\tau)]$ of (PFC)$_s$ starting from B converges uniformly in τ to the global attractor $\mathcal{A}_\tau$ of the periodic system (PFC)$_s$.

References

1. H. Brézis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, 1973.

2. V. V. Chepyzhov and M. I. Vishik, Attractors of non-autonomous dynamical systems and their dimension, J. Math. Pures Appl., **73** (1994), 279-333.

3. P. Colli and Ph. Laurençot, Weak solutions to the Penrose-Fife phase field model for a class of admissible heat flux laws, submitted.

4. P. Colli, Ph. Laurençot and J. Sprekels, Global solution to the Penrose-Fife phase field model with special heat flux laws, preprint.

5. P. Colli and J. Sprekels, On a Penrose-Fife model with zero interfacial energy leading to a phase-field system of relaxed Stefan type, Ann. Mat. Pura. Appl. (4), **169**(1995), 269-289.

6. A. Damlamian and N. Kenmochi, Evolution equations associated with non-isother-
mal phase transitions, pp. 62-77, in *Functional Analysis and Global Analysis*, ed. T. Sunada and P. W. Sy, Springer-Verlag, Singapore, 1997.

7. J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs **25**, Amer. Math. Soc., Providence, R. I., 1988.

8. A. Ito, N. Yamazaki and N. Kenmochi, Attractors of nonlinear evolution systems generated by time-dependent subdifferentials in Hilbert spaces, to appear in lecture notes in pure and applied mathematics, Marcel Dekker, New York.

9. A. Ito, N. Kenmochi and N. Yamazaki, Attractors of periodic systems generated by time-dependent subdifferentials, Technical Reports of Mathematical Science, Chiba Univ., Vol. 13 (1997), No. 12.

10. N. Kenmochi and M. Niezgódka, Systems of nonlinear parabolic equations for phase change problems, Adv. Math. Sci. Appl., **3**(1993/94), 89-117.

11. N. Kenmochi and M. Ôtani, Instability of periodic solutions of some evolution equations governed by time-dependent subdifferential operators, Proc. Japan Acad., **61**(1985), 4-7.

12. N. Kenmochi and M. Ôtani, Asymptotic behavior of periodic systems generated by time-dependent subdifferential operators, Funk. Ekvac., **29**(1986), 219-236.

13. Ph. Laurençot, Solutions to a Penrose-Fife model of phase-field type, J. Math. Anal. Appl., **185**(1994), 262-274.

14. K. Shirakawa, A. Ito, N. Yamazaki and N. Kenmochi, Asymptotic stability for evolution equations governed by subdifferentials, to appear in Gakuto International Series, Math. Sci. Appl., Gakkōtosho, Tokyo.

15. R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, Berlin, 1988.